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Classification of Solutions of Higher-Order Differential Equations and Inequalities with Deviating Arguments*

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INTRODUCTION

In this paper we classify solutions of certain differential equations and differential inequalities with deviating arguments. This is done under certain assumptions concerning the convergence, or divergence, of an improper integral over $[t_0, \infty)$ where the integrand involves certain coefficient functions. Much of our work relates directly to those in [1, 2, 4–6, 8, 9].

A nontrivial solution of a differential equation which exists on the half-line $[t_0, \infty)$ is said to be oscillatory if it has a zero on the interval $[t, \infty)$ for every $t > t_0$; otherwise it is said to be nonoscillatory on $[t_0, \infty)$. Throughout this paper, only nontrivial solutions which are extensible infinitely to the right of zero on the real line are considered.

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Consider the nth-order deviating argument differential equations of the form

$$y^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} F_i(t, y(t), y(g_i(t))) = 0, \tag{1.1}$$

where the functions F_i and g_i ($i = 1, 2,..., m$) satisfy the following conditions:

(H_1) *$F_i \in C([0, \infty) \times \mathbb{R}^2)$, $i = 1, 2,..., m$, and for some index k, $1 \leqslant k \leqslant m$, $F_k(t, u, v_k)$ is nondecreasing in u and in v_k for each fixed t;*

(H_2) *$F_i(t, u, v_i)$ has the same sign as that of u and v_i for $i = 1, 2,..., m$, if $uv_i > 0$;*

(H_3) *$g_i \in C[0, \infty)$, $g_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2,..., m$.*

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For convenience, we shall employ the following notation. S = the set of all solutions of the given differential equation which are continuable indefinitely to the right.

For an integer l, $0 \leqslant l \leqslant n-1$,

$$S_l^{+\infty} = \{y \in S\colon \lim_{t\to\infty} y^{(j)}(t) = \infty, j = 0, 1,\ldots, l\},$$

$$S_l^{-\infty} = \{y \in S\colon -y \in S_l^{+\infty}\},$$

$$S^e = \{y \in S\colon 0 < \lim_{t\to\infty} y(t) < \infty \text{ and } \lim_{t\to\infty} y^{(j)}(t) = 0, j = 1, 2,\ldots, n-1\},$$

$$S^{-e} = \{y \in S\colon -y \in S^e\},$$

$$S^0 = \{y \in S\colon \lim_{t\to\infty} y^{(j)}(t) = 0 \text{ monotonically}, j = 0, 1,\ldots, n-1\},$$

$$S^{\sim} = \{y \in S\colon y(t) \text{ is oscillatory}\}.$$

We shall provide three preliminary lemmas. The first two are those of Kiguradze [3].

LEMMA 1.1. [3, Lemma 1]. *If y is a function, which together with its derivatives of order up to $(n-1)$ inclusive, is absolutely continuous and of constant sign on the interval $[t_0, \infty)$ and $y^{(n)}(t)\,y(t) \leqslant 0$ on $[t_0, \infty)$, then there is an integer l, $0 \leqslant l \leqslant n-1$, which is odd when n is even and even when n is odd, such that*

$$\begin{aligned} y^{(j)}(t)\,y(t) &\geqslant 0, \qquad j = 0, 1,\ldots, l, \\ (-1)^{n+j-1}\,y^{(j)}(t)\,y(t) &\geqslant 0, \qquad j = l+1,\ldots, n, \end{aligned} \tag{1.2}$$

and

$$|\,y(t)| \geqslant \frac{(t-t_0)^{n-1}}{(n-1)(n-2)\cdots(n-l)}\,|\,y^{(n-1)}(2^{n-l-1}t)| \tag{1.3}$$

for $t \in [t_0, \infty)$.

LEMMA 1.2 [3, Lemma 2]. *If y is a function, which together with its derivatives of order up to $(n-1)$ inclusive, is absolutely continuous and of constant sign on the interval $[t_0, \infty)$ and $y^{(n)}(t)\,y(t) \geqslant 0$ on $[t_0, \infty)$, then either*

$$y^{(j)}(t)\,y(t) \geqslant 0, \qquad j = 0, 1,\ldots, n, \tag{1.4}$$

or there is an integer l, $0 \leqslant l \leqslant n-2$, which is even when n is even and odd n is odd, such that

$$\begin{aligned} y^{(j)}(t)\,y(t) &\geqslant 0, \qquad j = 0, 1,\ldots, l, \\ (-1)^{n+j}\,y^{(j)}(t)\,y(t) &\geqslant 0, \qquad j = l+1,\ldots, n, \end{aligned} \tag{1.5}$$

and the inequality (1.3) holds for $t \in [t_0, \infty)$.

LEMMA 1.3. *If y is as in* Lemma 1.1 (*or Lemma* 1.2), *and if for some* $0 \leqslant j \leqslant n-1$,

$$\lim_{t\to\infty} y^{(j)}(t) = c, \qquad c \in \mathbb{R},$$

then

$$\lim_{t\to\infty} y^{(j+m)}(t) = 0, \qquad m = 1, 2, \ldots, n-j-1.$$

Proof. If y is as in Lemma 1.1, the proof is referred to that of [7, Lemma 2]. A similar proof applies to the case in which y is as in Lemma 1.2.

The following result generalizes Ladde's result [6, Theorem 2.1] to the arbitrary order delay differential equation (1.1). A related theorem for (1.1) with odd $n \geqslant 3$ is provided by Theorem 1.3. Further related results for Eq. (1.1′), which appears after Theorem 1.4, are provided by Theorems 1.5 and 1.7. For a comparison of the results of this section it is natural to compare Theorem 1.1 to Theorem 1.7 and Theorem 1.3 to Theorem 1.5.

THEOREM 1.1. *In* (1.1) *let $n \geqslant 2$ be even and let F_k satisfy the following condition*

$$\int^{\infty} F_k(t, \gamma t, \gamma g_k(t))\, dt = \pm\infty \qquad \textit{for each constant} \quad \gamma \neq 0. \tag{1.6}$$

Then

$$S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^{e} \cup S^{-e} \cup S^{0} \cup S^{\sim}.$$

Proof. Let $y \in S - S^{\sim}$. Then there is a $T_0 > 0$ such that $y(t) > 0$ on $[T_0, \infty)$ or $y(t) < 0$ on $[T_0, \infty)$.

Case 1. Let $y(t) > 0$ on $[T_0, \infty)$. There is a $T_1 \geqslant T_0$ such that $g_i(t) \geqslant T_0$ for $t \geqslant T_1$ ($i = 1, 2, \ldots, m$). By (1.1) and (H_2), $y^{(n)}(t) > 0$ for $t \geqslant T_1$. There is a $T_2 \geqslant T_1$ such that each $y^{(j)}(t)$, $j = 0, 1, \ldots, n-1$, is of constant sign for $t \geqslant T_2$.

If $y'(t) > 0$ for $t \geqslant T_2$, then by Lemma 1.2, $y''(t) > 0$. It follows that $y(t) \to \infty$ as $t \to \infty$ and we have

$$\lim_{t\to\infty} \frac{y(t)}{t} = \lim_{t\to\infty} \frac{y(t) - y(T_2)}{t - T_2} \geqslant y'(T_2) > 0.$$

Let $y'(T_2) = 2\beta$. Then there is a $T_3 \geqslant T_2$ such that $t^{-1}y(t) > \beta$ for $t \geqslant T_3$. There is a $T_4 \geqslant T_3$ such that $g_k(t) \geqslant T_3$ for $t \geqslant T_4$. Thus we have

$$y(t) > \beta t \qquad \text{and} \qquad y(g_k(t)) > \beta g_k(t) \qquad \text{for} \quad t \geqslant T_4. \tag{1.7}$$

Integrating (1.1) from T_4 to $t > T_4$ and using (1.7), (H_1), and (1.6), we have

$$y^{(n-1)}(t) \geqslant y^{(n-1)}(T_4) + \int_{T_4}^{t} F_k(s, \beta s, \beta g_k(s))\, ds \to \infty \qquad \text{as} \quad t \to \infty.$$

This says that (1.4) holds, since (1.5) cannot hold for $j = n - 1$. Thus $y^{(j)}(t) \to \infty$ as $t \to \infty$ for all $0 \leqslant j \leqslant n-1$. Consequently $y \in S_{n-1}^{+\infty}$.

If $y'(t) < 0$ for $t \geqslant T_2$, then $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y'(\infty) = y''(\infty) = \cdots = y^{(n-1)}(\infty) = 0$. That is, $y \in S^e \cup S^0$.

Case 2. If $y(t) < 0$ for $t \geqslant T_0$, then the proof follows similar to Case 1 and we have the conclusion that $S = S_{n-1}^{-\infty} \cup S^{-e} \cup S^0$ for this case.

We can modify the proof of Theorem 1.1 to show that Theorem 1.1 still holds for the equation

$$y^{(n)}(t) - \sum_{i=1}^{m} F_i(t, y(t), y(g_i(t)), y'(t), y'(g_i(t))) = 0$$

if we replace conditions (H_1), (H_2), and (1.6), respectively, by

(H_1) $F_i \in C([0, \infty) \times \mathbb{R}^4)$, $i = 1, 2, \ldots, m$, *and for some index* k, $1 \leqslant k \leqslant m$, $F_k(t, u, u_k, v, v_k)$ *is nondecreasing in* u *and in* u_k, *and is also nondecreasing in* v *and in* v_k *when* $uu_k > 0$ *and* $vv_k > 0$;

(H_2') $F_i(t, u, u_i, v, v_i)$ *has the same sign as that of* u *and* u_i *for* $i = 1, 2, \ldots, m$, *if* $uu_i > 0$; *and*

$$\int^{\infty} F_k(t, \gamma t, \gamma g_k(t), \alpha, \alpha) = \pm\infty \qquad \text{if} \quad \alpha\gamma > 0. \tag{1.6'}$$

The next result generalizes Theorem 2.1 of Laddas, Ladde, and Papadakis [4] to Eq. (1.1) for even n. It also includes Theorem 3.1 of Ladas, Lakshmikantham, and Papadakis [5]. The proof of our theorem is obtained by modifying that of the previous mentioned result where only linear equations are considered. At the conclusion of Section 2 we have further comments relating to [5, Theorem 3.1] and the result to follow. Further related results for Eq. (1.1) with odd $n \geqslant 3$ and for equation (1.1') are provided by Theorems 1.4, 1.6, and 1.8. It is natural to compare the conclusion of Theorem 1.2 to Theorem 1.8 and Theorem 1.4 to Theorem 1.6, though we point out that the assumptions of Theorems 1.2 and 1.4 involve condition (1.8) rather than condition (1.14) used in the other results.

THEOREM 1.2. *In* (1.1) *let* $n \geqslant 2$ *be even and let* F_k *satisfy* (1.6). *Furthermore, assume that*

$$\begin{aligned} F_k(t, \gamma w, \gamma g_k(w)) &\leqslant w F_k(t, \gamma, \gamma) \qquad \textit{for any} \quad \gamma > 0, \\ &\geqslant w F_k(t, \gamma, \gamma) \qquad \textit{for any} \quad \gamma < 0, \end{aligned} \tag{1.8}$$

for all sufficiently large t *and* $w > 0$. *Then* $S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^0 \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.1, there is a $T_2 > 0$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n-1$, is of constant sign for $t \geqslant T_2$.

Case 1. Let $y(t) > 0$ for $t \geqslant T_2$. If $y'(t) > 0$ for $t \geqslant T_2$, then it follows from the proof of Theorem 1.1 that $y \in S_{n-1}^{+\infty}$. If $y'(t) < 0$ for $t \geqslant T_2$, then $\lim_{t\to\infty} y(t)$ exists and is nonnegative. We claim that $y(t) \to 0$ as $t \to \infty$. Otherwise suppose $y(t) \to 2\beta > 0$ as $t \to \infty$. By Lemma 1.3, $y'(\infty) = y''(\infty) = \cdots = y^{(n-1)}(\infty) = 0$. Since $y'(t) < 0$ for $t \geqslant T_2$, by (1.5) of Lemma 1.2, $(-1)^j y^{(j)}(t) > 0$, $j = 0, 1,..., n$, for $t \geqslant T_2$. There is a $T_3 \geqslant T_2$ such that

$$y(g_k(t)) \geqslant \beta \qquad \text{and} \qquad y(t) \geqslant \beta \qquad \text{for} \quad t \geqslant T_3. \tag{1.9}$$

Integrating (1.1) from T_3 to $t > T_3$, we have

$$y^{(n-1)}(t) = y^{(n-1)}(T_3) + \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds. \tag{1.10}$$

Let $t \to \infty$, we obtain

$$y^{(n-1)}(T_3) = -\int_{T_3}^{\infty} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds. \tag{1.11}$$

Integrating (1.10) from T_3 to $t > T_3$ and using (H_2), (H_3), (1.11), (1.9), and (1.10), we have

$$\begin{aligned}
y^{(n-2)}(t) &= y^{(n-2)}(T_3) + (t - T_3)\, y^{(n-1)}(T_3) \\
&\quad + \int_{T_3}^{t} (t-s) \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds \\
&= y^{(n-2)}(T_3) + \int_{T_3}^{t} (T_3 - s) \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds \\
&\quad - (t - T_3) \int_{t}^{\infty} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds \\
&\leqslant y^{(n-2)}(T_3) + T_3 \int_{T_3}^{t} y^{(n)}(s)\, ds - \int_{T_3}^{t} s \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds \\
&\leqslant y^{(n-2)}(T_3) - T_3 y^{(n-1)}(T_3) - \int_{T_3}^{t} sF_k(s, \beta, \beta)\, ds \\
&\leqslant y^{(n-2)}(T_3) - T_3 y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s, \beta s, \beta g_k(s))\, ds.
\end{aligned}$$

By (1.6), $y^{(n-2)}(t)$ would be negative for all sufficiently large t, which is a contradiction. Therefore, $y(t) \to 0$ as $t \to \infty$. Moreover, by Lemma 1.3, $y \in S^0$.

Case 2. Let $y(t) < 0$ for $t \geqslant t_0$. Then the proof follows similar to Case 1 and it concludes that $y \in S_{n-1}^{-\infty} \cup S^0$ for this case.

THEOREM 1.3. *In* (1.1) *let* $n \geqslant 3$ *be odd and let* F_k *satisfy* (1.6). *Then* $S = S^e \cup S^{-e} \cup S^0 \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.1, we discuss the following two cases.

Case 1. Let $y(t) > 0$ for $t \geqslant T_0$. There is a $T_1 \geqslant T_0$ such that $g_i(t) \geqslant T_0$ for $t \geqslant T_1$ $(i = 1, 2,..., m)$. By (1.1) and (H_2), $y^{(n)}(t) < 0$ for $t \geqslant T_1$. There is a $T_2 \geqslant T_1$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n-1$, is of constant sign for $t \geqslant T_2$.

If $y'(t) > 0$ for $t \geqslant T_2$, then $l \geqslant 2$ in (1.2) of Lemma 1.1. Thus as in the proof of Theorem 1.1 there is a $\beta > 0$ and a $T_3 \geqslant T_2$ such that

$$y(t) \geqslant \beta t \qquad \text{and} \qquad y(g_k(t)) \geqslant \beta g_k(t) \qquad \text{for} \quad t \geqslant T_3. \tag{1.12}$$

Integrating (1.1) from T_3 to $t > T_3$ and using (1.12) and (H_1), we have

$$y^{(n-1)}(t) \leqslant y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s, \beta s, \beta g_k(s))\, ds.$$

By (1.6), $y^{(n-1)}(t)$ would be negative for all sufficiently large t, which is a contradiction. Therefore, $y'(t) < 0$ for $t \geqslant T_2$. It follows that $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$, $j = 1, 2,..., n-1$. That is, $y \in S^e \cup S^0$.

Case 2. Let $y(t) < 0$ for $t \geqslant T_0$. Then the proof follows, similar to Case 1, that $y \in S^{-e} \cup S^0$.

THEOREM 1.4. *In* (1.1) *let* $n \geqslant 3$ *be odd and let* F_k *satisfy* (1.6). *Furthermore, assume that* (1.8) *holds. Then* $S = S^0 \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.1, we have the following two cases.

Case 1. If $y(t) > 0$ for $t \geqslant T_0$, then as in the proof of Theorem 1.3, we have the conclusion that $y'(t) < 0$ for $t \geqslant T_2$. Then $\lim_{t\to\infty} y(t)$ exists and is nonnegative. Moreover, by Lemma 1.3, $y^{(j)}(t) \to 0$ as $t \to \infty$, $j = 1, 2,..., n-1$. We claim that $y(t) \to 0$ as $t \to \infty$. Otherwise suppose that $y(t) \to 2\beta > 0$ as $t \to \infty$. Then there exists a $T_3 \geqslant T_2$ such that

$$y(g_k(t)) > \beta \qquad \text{and} \qquad y(t) > \beta \qquad \text{for} \quad t \geqslant T_3.$$

Since $y'(t) < 0$ for $t \geqslant T_2$, by (1.2) of Lemma 1.1, $(-1)^j y^{(j)}(t) > 0, j = 0, 1,..., n$, for $t \geqslant T_2$. Integrating (1.1) from T_3 to $t > T_3$, we have

$$y^{(n-1)}(t) = y^{(n-1)}(T_3) - \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds. \tag{1.13}$$

Let $t \to \infty$; we obtain

$$y^{(n-1)}(T_3) = \int_{T_3}^{\infty} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds.$$

Integrating (1.13) from T_3 to $t > T_3$ and using the same procedures as in the proof of Theorem 1.2, we have

$$y^{(n-2)}(t) \geqslant y^{(n-2)}(T_3) - T_3 y^{(n-1)}(T_3) + \int_{T_3}^{t} F_k(s, \beta s, \beta g_k(s))\, ds,$$

which would be positive for all sufficiently large t, a contradiction. Therefore, $y \in S^0$.

Case 2. If $y(t) < 0$ for $t \geqslant T_0$, then the proof follows similar to Case 1.

We now consider the equation

$$y^{(n)}(t) + (-1)^n \sum_{i=1}^{m} F_i(t, y(t), y(g_i(t))) = 0, \tag{1.1'}$$

where F_i, g_i ($i = 1, 2,..., m$) satisfy (H_1), (H_2), and (H_3). We define the following subsets of S:

$$S^E = \{y \in S: 0 < \lim_{t\to\infty} y(t) \leqslant \infty \text{ and } \lim_{t\to\infty} y^{(j)}(t) = 0, j = 1, 2,..., n-1\},$$

$$S^{-E} = \{y \in S: -y \in S^E\}.$$

Clearly, $S^e \subseteq S^E$ and $S^{-e} \subseteq S^{-E}$ hold.

We have the following theorem corresponding to Theorem 1.1.

THEOREM 1.5. *In* (1.1') *let* $n \geqslant 2$ *be even and let* F_k *satisfy* (1.6). *Then* $S = S^E \cup S^{-E} \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. Then there is a $T_0 > 0$ such that $y(t) > 0$ on $[T_0, \infty)$ or $y(t) < 0$ on $[T_0, \infty)$.

Case 1. Let $y(t) > 0$ on $[T_0, \infty)$. There is a $T_1 \geqslant T_0$ such that $g_i(t) \geqslant T_0$ for $t \geqslant T_1$ ($i = 1, 2,..., m$). By (1.1') and (H_2), $y^{(n)}(t) < 0$ for $t \geqslant T_1$. There is a $T_2 \geqslant T_1$ such that each $y^{(j)}(t), j = 0, 1,..., n-1$, is of constant sign for $t \geqslant T_2$. By Lemma 1.1, $y'(t) > 0$ for $t \geqslant T_2$. It follows that $0 < \lim_{t\to\infty} y(t) \leqslant \infty$. Now either $y''(t) > 0$ or $y''(t) < 0$ for $t \geqslant T_2$ and $0 \leqslant \lim_{t\to\infty} y'(t) \leqslant \infty$. In either case, either $y'(t) \to 0$ as $t \to \infty$, and we are done, or $\lim_{t\to\infty} y'(t) > 0$. The remainder of the argument is much as in the proof of Theorems 1.1 and 1.3. That is, $y \in S^E$.

Case 2. If $y(t) < 0$ for $t \geqslant T_0$, then the proof follows similar to Case 1 and we have the conclusion that $S = S^{-E}$ for this case.

THEOREM 1.6. *In* (1.1′) *let* $n \geqslant 2$ *be even and let* F_k *satisfy the condition*

$$\int^{\infty} F_k(t, \gamma, \gamma)\, dt = \pm\infty \qquad \textit{for each} \quad \gamma \neq 0. \tag{1.14}$$

Then $S = S^{\sim}$.

Proof. Let y be a nonoscillatory solution of (1.1′). We can assume that $y(t) > 0$ for $t \geqslant T_0$. As in the proof of Theorem 1.5, there is a $T_2 \geqslant T_0$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n-1$, is of constant sign and $y'(t) > 0$ for $t \geqslant T_2$. It follows that $y(t) > \beta$ for some $\beta > 0$ and for each $t \geqslant T_2$. There is a $T_3 \geqslant T_2$ such that $y(g_k(t)) > \beta$ for $t \geqslant T_3$. Integrating (1.1′) from T_3 to $t > T_3$, we have

$$\begin{aligned} y^{(n-1)}(t) &= y^{(n-1)}(T_3) - \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds \\ &\leqslant y^{(n-1)}(T_3) - \int_{T_3}^{t} F_k(s, \beta, \beta)\, ds. \end{aligned}$$

It follows from (1.14) that $y^{(n-1)}(t)$ would be negative for all sufficiently large t. This is a contradiction, since by Lemma 1.1, $y^{(n-1)}(t) > 0$ for $t \geqslant T_2$.

THEOREM 1.7. *In* (1.1′) *let* $n \geqslant 3$ *be odd and let* F_k *satisfy* (1.6). *Then* $S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^E \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. There is a $T_0 > 0$ such that $y(t) > 0$ or $y(t) < 0$ on $[T_0, \infty)$.

Case 1. Let $y(t) > 0$ on $[T_0, \infty)$. There is a $T_1 \geqslant T_0$ such that $g_i(t) \geqslant T_0$, $i = 1, 2,..., m$, for $t \geqslant T_1$. Then $y^{(n)}(t) > 0$ on $[T_1, \infty)$. There is a $T_2 \geqslant T_1$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n-1$, is of constant sign on $[T_2, \infty)$. By Lemma 1.2, either (1.4) holds or there is an odd integer l such that (1.5) holds. If (1.4) holds, then by an argument similar to a portion of the proof of Theorem 1.1 it follows that $y^{(n-1)}(t) \to \infty$ as $t \to \infty$ and then $y \in S_{n-1}^{+\infty}$. If (1.5) holds, then $y'(t) > 0$ and either $y''(t) > 0$ or $y''(t) < 0$ on $[T_2, \infty)$.

It follows that $0 \leqslant \lim_{t\to\infty} y'(t) \leqslant \infty$. If $y'(t) \to 0$ as $t \to \infty$, then by Lemma 1.3, $y \in S^E$. If $0 < \lim_{t\to\infty} y'(t) \leqslant \infty$, then as previously we have $y^{(n-1)}(t) > 0$ for all sufficiently large t, a contradiction. Thus, $y \in S_{n-1}^{+\infty} \cup S^E$.

Case 2. Let $y(t) < 0$ on $[T_0, \infty)$. Then by an similar argument similar to that of Case 1, we can show that $y \in S_{n-1}^{-\infty} \cup S^{-E}$.

THEOREM 1.8. *In* (1.1′) *let* $n \geqslant 3$ *be odd and let* F_k *satisfy* (1.14). *Then* $S = S_{n-1}^{+\infty} \cup S_{n-1}^{-\infty} \cup S^{\sim}$.

Proof. Let $y \in S - S^{\sim}$. As in the proof of Theorem 1.7, we have $y \in S_{n-1}^{+\infty}$ or $y'(t) > 0$ and $y^{(n-1)}(t) < 0$ on $[T_2, \infty)$ for Case 1. If $y'(t) > 0$ on $[T_2, \infty)$, then there is a $T_3 \geqslant T_2$ and a $\beta > 0$ such that $y(t) > \beta$ and $y(g_k(t)) > \beta$ for $t \geqslant T_3$. It follows that

$$y^{(n-1)}(t) = y^{(n-1)}(T_3) + \int_{T_3}^{t} \sum_{i=1}^{m} F_i(s, y(s), y(g_i(s)))\, ds$$

$$\geqslant y^{(n-1)}(T_3) + \int_{T_3}^{t} F_k(s, \beta, \beta)\, ds.$$

By (1.14), $y^{(n-1)}(t)$ would be positive for all sufficiently large t, a contradiction. Thus, $y \in S_{n-1}^{+\infty}$.

Similarly, we have $y \in S_{n-1}^{-\infty}$ for Case 2.

2

Consider the deviating argument differential inequalities of the forms

$$y(t)[y^{(n)}(t) + F(t, y(t), y(g(t)))] \leqslant 0 \tag{2.1}$$

and

$$y(t)[y^{(n)}(t) - F(t, y(t), y(g(t)))] \geqslant 0, \tag{2.2}$$

where $F(t, u, v)$ satisfies (H_1), (H_2), for $F = F_k$ and $g(t) \to \infty$ as $t \to \infty$.

We shall investigate the oscillatory behavior of solutions of inequalities (2.1) and (2.2) via the oscillatory behavior of solutions of a class of differential inequalities

$$x(t)[x^{(n)}(t) + p(t)\, N(x(t))] \leqslant 0 \tag{2.3}$$

and

$$x(t)[x^{(n)}(t) - p(t)\, N(x(t))] \geqslant 0, \tag{2.4}$$

where p is nonnegative and continuous on $[0, \infty)$, and $N \in C(-\infty, \infty)$ is nondecreasing on $[0, \infty)$ and $xN(x) > 0$ for $x \neq 0$. Corollaries 2.1 and 2.2, under weaker conditions, extend Teufel's results for bounded solutions in [8] to arbitrary even-order delay differential equations. Corollary 2.3 reduces to Chiou's Theorem 2.7 [1] and also improves Erbe's Theorem 3.1 [2] and the sufficient part of Erbe's Theorem 3.2 [2] in the case of $n = 2$. Erbe's Corollary 2.5 [2] and Chiou's Theorem 2.1 [1] are special cases of Corollary 2.3 for $n = 2$ and $F(t, y(t), y(g(t))) = p(t) f(y(t), y(g(t)))$. For $n = 2$ and $g(t) = t$, Corollary 2.3 also includes the sufficient part of Chiou's Theorem 1 [9].

THEOREM 2.1. *Let x be a nonoscillatory solution of* (2.3), *let l satisfy* (1.2), *and let*

$$\int^{\infty} t^{n-l} p(t)\, N(\gamma t^{l-1})\, dt = \pm\infty \qquad \text{for each} \quad \gamma \neq 0. \tag{2.5}$$

Then, if n is even, $x^{(l-1)}(t)$ tends to infinity monotonically. If n is odd, either $x(t)$ tends to a limit or $x^{(l-1)}(t)$ tends to infinity monotonically.

Proof. Suppose that $x(t) > 0$ on $[T_0, \infty)$. It follows that

$$x^{(n)}(t) + p(t)\, N(x(t)) < 0 \qquad \text{for} \quad t \in [T_0, \infty). \tag{2.6}$$

Thus $x^{(n)}(t) \leqslant 0$ on $[T_0, \infty)$ follows also. There is a $T_1 \geqslant T_0$ such that each $x^{(j)}(t), j = 0, 1,..., n-1$, is of constant sign on $[T_1, \infty)$. If n is even, then by Lemma 1.1, there is an odd integer l such that (1.2) holds. Since $x^{(l-1)}(t) > 0$ and $x^{(l)}(t) > 0$ on $[T_1, \infty)$, there is an $L > 0$ such that $x^{(l-1)}(t) > L$ for $t \geqslant T_1$. Integrating this repeatedly and using (1.2), there exist $\gamma > 0$ and $T_2 \geqslant T_1$ such that

$$x(t) > \gamma t^{l-1} \qquad \text{for} \quad t \geqslant T_2. \tag{2.7}$$

Integrating (2.6) from $s \geqslant T_2$ to $t > s$, we obtain

$$0 \leqslant x^{(n-1)}(t) \leqslant x^{(n-1)}(s) - \int_s^t p(u)\, N(x(u))\, du$$

or

$$x^{(n-1)}(\sigma) \geqslant \int_\sigma^t p(u)\, N(x(u))\, du \qquad \text{for} \quad t > \sigma \geqslant T_2.$$

Integrating this from $s \geqslant T_2$ to $t > s$, we have

$$0 \geqslant x^{(n-2)}(t) \geqslant x^{(n-2)}(s) + \int_s^t (u - s)\, p(u)\, N(x(u))\, du$$

or

$$x^{(n-2)}(\sigma) \leqslant - \int_\sigma^t (u - \sigma)\, p(u)\, N(x(u))\, du \qquad \text{for} \quad t > \sigma \geqslant T_2.$$

Continuing this process, we get

$$x^{(l)}(\sigma) \geqslant \int_\sigma^t \frac{(u-\sigma)^{n-l-1}}{(n-l-1)!} p(u)\, N(x(u))\, du \qquad \text{for} \quad t > \sigma \geqslant T_2 .$$

Integrating this from T_2 to $t > T_2$ and using (2.7), we obtain

$$x^{(l-1)}(t) \geqslant x^{(l-1)}(T_2) + \int_{T_2}^t \frac{(u-T_2)^{n-l}}{(n-l)!} p(u)\, N(\gamma u^{l-1})\, du.$$

It follows from (2.5) that $x^{(l-1)}(t) \to \infty$ as $t \to \infty$.

If n is odd, then l is an even integer satisfying (1.2). If $l = 0$, then $x'(t) < 0$ for all sufficiently large t and then $x(t)$ tends to a limit. If $l \geqslant 2$, then the proof for n even go through.

Let x be a nonoscillatory solution of (2.3) with n even. If x is bounded and assume that $x(t) > 0$ on $[T_0, \infty)$, then $l = 1$ in Lemma 1.1. Thus we have the following

COROLLARY 2.1. *If* (2.5) *holds for* $l = 1$, *then every bounded solution of* (2.3) *with n even is oscillatory.*

THEOREM 2.2. *Let* $0 < \liminf_{t\to\infty} (g(t)/t) \leqslant 1$. *Assume that y is a nonoscillatory solution of* (2.1), *l satisfies* (1.2), *and* (2.5) *holds. Furthermore, assume that*

$$uF(t, u, u) \geqslant up(t)\, N(u) \qquad \textit{for} \quad u \neq 0. \tag{2.8}$$

Then, if n is even, $y^{(l-1)}(t)$ tends to infinity monotonically. If n is odd, either $y(t)$ tends to a limit or $y^{(l-1)}(t)$ tends to infinity monotonically.

Proof. As in the proof of Theorem 2.1, we only consider the case when n is even. Without loss of generality, we can assume that $y(t) > 0$ on $[T_0, \infty)$. There is a $T_1 \geqslant T_0$ such that $y(g(t)) > 0$ for $t \geqslant T_1$. It follows from (2.1) that $y^{(n)}(t) < 0$ on $[T_1, \infty)$. There is a $T_2 \geqslant T_1$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n-1$, is of constant sign on $[T_2, \infty)$. Let l be as in Lemma 1.1 and assume that $y^{(l-1)}(t)$ is bounded. Then $\infty > \lim_{t\to\infty} y^{(l-1)}(t) = L > 0$. There is a $T_3 \geqslant T_2$ such that

$$\tfrac{1}{2} L t^{l-1} < y(t) < L t^{l-1} \qquad \text{for} \quad t \geqslant T_3 . \tag{2.9}$$

Thus we have

$$y(g(t)) \geqslant \tfrac{1}{2} L (g(t))^{l-1} \geqslant \tfrac{1}{2} L (g(t)/t)^{l-1}\, t^{l-1}$$

for all sufficiently large t. Let $\liminf_{t\to\infty} (g(t)/t) = \alpha > 0$, then we have

$$y(g(t) \geqslant \tfrac{1}{2} L (\tfrac{1}{2}\alpha)^{l-1}\, t^{l-1} \geqslant \tfrac{1}{2}(\tfrac{1}{2}\alpha)^{l-1} y(t) = My(t) \tag{2.10}$$

for all sufficiently large t. It follows that

$$0 \geqslant y^{(n)}(t) + F(t, y(t), y(g(t))) \geqslant y^{(n)}(t) + F(t, My(t), My(t))$$
$$\geqslant y^{(n)}(t) + p(t)\, N(My(t)).$$

Replacing $N(x)$ by $N(Mx)$ in Theorem 2.1, we get a contradiction.

Similar to Corollary 2.1, we have the following

COROLLARY 2.2. *If all the hypotheses of Theorem 2.2 hold and $l = 1$ in* (2.5), *then every bounded solution of* (2.1) *with n even is oscillatory.*

In Corollary 2.2, we can remove the condition $0 < \liminf_{t\to\infty} (g(t)/t) \leqslant 1$ and only require that $\lim_{t\to\infty} g(t) = \infty$. That is because in the proof of Theorem 2.2, we have $y(t) \to L$ as $t \to \infty$ and then corresponding to (2.10) we have $y(g(t)) \geqslant \frac{1}{2}L \geqslant \frac{1}{2} y(t)$ for all sufficiently large t. The proof of Theorem 2.2 goes through if we replace M by $\frac{1}{2}$.

THEOREM 2.3. *Let* $0 < \liminf_{t\to\infty} (g(t)/t) \leqslant 1$. *Assume that y is a non-oscillatory solution of* (2.1), *l satisfies* (1.2) *and*

$$\int^{\infty} t^{n-l} F(t, \gamma t^{l-1}, \gamma t^{l-1})\, dt = \pm\infty \qquad \textit{for each} \quad \gamma \neq 0. \tag{2.11}$$

Then, if n is even, $y^{(l-1)}(t)$ tends to infinity monotonically. If n is odd, either $y(t)$ tends to a limit or $y^{(l-1)}(t)$ tends to infinity monotonically.

Proof. We can assume that $y(t) > 0$ for all sufficiently large t. As in the proof of Theorem 2.1, we only consider n even. Then there is a $T_1 > 0$ such that each $y^{(j)}(t)$, $j = 0, 1,..., n - 1$, is of constant sign on $[T_1, \infty)$ and an odd integer l satisfying (1.2). Since $y^{(l-1)}(t) > 0$ for $t \geqslant T_1$, $0 < \lim_{t\to\infty} y^{(l-1)}(t) \leqslant \infty$. If $y^{(l-1)}(t) \to \infty$ as $t \to \infty$, then our proof is done. Let $0 < \lim_{t\to\infty} y^{(l-1)}(t) = L < \infty$. As in the proof of Theorem 2.2, (2.9) and (2.10) hold.

Let $p(t) = (1/Lt^{l-1}) F(t, \frac{1}{2}L(\frac{1}{2}\alpha)^{l-1} t^{l-1}, \frac{1}{2}L(\frac{1}{2}\alpha)^{l-1} t^{l-1})$ and let $N(x) = x$ in Theorem 2.1. Then, by (2.9) and (2.10), we can verify that $y(t)$ satisfies (2.3). Moreover, we have

$$\int^{\infty} t^{n-l} p(t)\, N(Lt^{l-1})\, dt = \int^{\infty} t^{n-l} F(t, \tfrac{1}{2}L(\tfrac{1}{2}\alpha)^{l-1} t^{l-1}, \tfrac{1}{2}L(\tfrac{1}{2}\alpha)^{l-1} t^{l-1})\, dt = \infty.$$

By Theorem 2.1, we get a contradiction.

Similar to Corollary 2.2, we have the following

COROLLARY 2.3. *If* $\int^{\infty} t^{n-1} F(t, \gamma, \gamma)\, dt = \pm\infty$ *for each* $\gamma \neq 0$, *then every bounded solution of* (2.1) *with n even is oscillatory.*

Applying the same argument as in the proofs of Theorems 2.1, 2.2, and 2.3, we have the following theorems.

THEOREM 2.4. *Let x be a nonoscillatory solution of* (2.4), *l satisfies* (1.5) *with $l < n$ and* (2.5) *holds. Then, if n is odd, $x^{(l-1)}(t)$ tends to infinity monotonically. If n is even, either $x(t)$ tends to a limit or $x^{(l-1)}(t)$ tends to infinity monotonically.*

THEOREM 2.5. *Let* $0 < \liminf_{t\to\infty} (g(t)/t) \leqslant 1$. *Assume that y is a nonoscillatory solution of* (2.2), *l satisfies* (1.5) *with $l < n$, and* (2.5) *holds. Furthermore, assume that* (2.8) *holds. Then, if n is odd, $y^{(l-1)}(t)$ tends to infinity monotonically. If n is even, either $y(t)$ tends to a limit or $y^{(l-1)}(t)$ tends to infinity monotonically.*

THEOREM 2.6. *Let* $0 < \liminf_{t\to\infty} (g(t)/t) \leqslant 1$. *Assume that y is a nonoscillatory solution of* (2.2), *l satisfies* (1.5) *with $l < n$, and* (2.11) *holds. Then, if n is odd, $y^{(l-1)}(t)$ tends to infinity monotonically. If n is even, either $y(t)$ tends to a limit or $y^{(l-1)}(t)$ tends to infinity monotonically.*

Equations (1.1) and (1.1′) in Section 1 are special cases of (2.1) and (2.2). In fact, most results of Section 1 could have been stated in terms of differential inequalities. One reason for not doing so is to present the readers a variety of settings in which the computations can be made. If we define as in [5], $S^{+\infty} = \bigcup_{l=1}^{n-1} S_l^{+\infty}$ and $S^{-\infty} = \bigcup_{l=1}^{n-1} S_l^{-\infty}$, the decomposition in [5, Theorem 3.1] is $S = S^{+\infty} \cup S^{-\infty} \cup S^0 \cup S^{\sim}$. Our Theorem 2.6 ($n$ even) has the same decomposition, where we give weaker condition on $g(t)$ and replace Eq. (3.1) of [5] by (2.11). Since, when $F(t, y(t), y(g(t))) = p(t)\, y(g(t))$, (2.11) becomes

$$\int^{\infty} t^{n-1} p(t)\, dt = \infty, \tag{2.12}$$

(2.11) is also weaker than Eq. (3.1) of [5]. If N has certain growth conditions, for instance, if $N(x)/x$ is monotone increasing, then (2.5) also may be replaced by (2.12).

Note added in proof. Some related results may be found in Bogar's paper [Oscillation of nth order differential equations with retarded argument, *SIAM J. Math. Anal.* **5** (1974), 473–481].

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